The Coinductive Formulation of Common Knowledge

Colm Baston and Venanzio Capretta

Functional Programming Lab
School of Computer Science
University of Nottingham

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Epistemic Modal Logic

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- **Knowledge generalisation**: states that the agent can derive all tautologies.
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- **Axiom T**: states that the agent’s knowledge must actually be true, distinguishing knowledge from belief or opinion.
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A propositional logic extended with a “knows” operator, written $K$, that represents the knowledge of an agent. The modal logic $S5$ provides rules for reasoning with knowledge operators:

- Knowledge generalisation states that the agent can derive all tautologies.
- Axiom $K$ states that the agent can follow implications, applying modus ponens to what they know.
- Axiom $T$ states that the agent’s knowledge must actually be true, distinguishing knowledge from belief or opinion.
- Axioms 4 and 5 state that the agent can perform introspection, knowing what they do and do not know.
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- Two states are related by an agent’s knowledge relation iff they cannot distinguish one state from the other based on their knowledge.
- The equivalence properties correspond to the S5 axioms:
  - Axiom T ↔ Reflexivity
  - Axiom 4 ↔ Transitivity
  - Axiom 5 ↔ Transitivity and Symmetry
We embed epistemic logic in type theory along these lines by postulating a set of states, and defining events as predicates over them:

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\text{State} : \text{Set} \quad \quad \text{Event} = \text{State} \rightarrow \text{Set}
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Shallow Embedding

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_\sqcap_{-} & : \text{Event} \rightarrow \text{Event} \rightarrow \text{Event} \\
e_1 \sqcap e_2 & = \lambda w. e_1 w \land e_2 w \\
_\sqcup_{-} & : \text{Event} \rightarrow \text{Event} \rightarrow \text{Event} \\
e_1 \sqcup e_2 & = \lambda w. e_1 w \lor e_2 w \\
_\sqsubseteq_{-} & : \text{Event} \rightarrow \text{Event} \rightarrow \text{Event} \\
e_1 \sqsubseteq e_2 & = \lambda w. e_1 w \rightarrow e_2 w \\
\sim_{-} & : \text{Event} \rightarrow \text{Event} \\
\sim e & = \lambda w. \neg(e \, w)
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\_ \cap \_ &: \text{Event} \rightarrow \text{Event} \rightarrow \text{Event} \\
{e_1 \cap e_2} &= \lambda w. e_1 w \land e_2 w \\
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{e_1 \cup e_2} &= \lambda w. e_1 w \lor e_2 w \\
\_ \sqsubset \_ &: \text{Event} \rightarrow \text{Event} \rightarrow \text{Event} \\
{e_1 \sqsubset e_2} &= \forall w. e_1 w \rightarrow e_2 w \\
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\_ \equiv \_ &: \text{Event} \rightarrow \text{Event} \rightarrow \text{Set} \\
{e_1 \equiv e_2} &= (e_1 \subset e_2) \land (e_2 \subset e_1) \\
\forall \_ &: \text{Event} \rightarrow \text{Set} \\
\forall e &= \forall w. e w
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Knowledge Operators

With these connectives, we can directly interpret the rules of S5 to define the concept of a knowledge operator:

Record $K_{Op} (K : \text{Event} \rightarrow \text{Event}) : \text{Set}$

Generalisation:

$\forall e \rightarrow \forall K e \in \text{axiom}_K : K (e_1 \sqsubseteq e_2) \subset (K e_1 \sqsubseteq K e_2)$

Axiom $T$:

$K e \subset e$

Axiom $4$:

$K e \subset K (K e)$

Axiom $5$:

$\neg K e \subset K (\neg K e)$

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An event family indexed by some type $X$ is a function:

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We can map a knowledge operator \( K \) onto the whole family by applying it to every member. We just write this as \( K E \):

\[
K E := \lambda x. K (E x)
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We say that $K$ preserves semantic entailment iff for every event family $E$ and event $e$:

$$\bigcap E \subset e \rightarrow \bigcap(K E) \subset K e$$
Preservation of Semantic Entailment

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- For knowledge generalisation, observe that $\forall e$ is equivalent to semantic entailment from the empty family: $\bigcap \emptyset \subset e$.

- For Axiom $K$, choose a “modus ponens” family indexed by the Booleans: $\bigcap \{g_1 \sqsubseteq e_2, e_1\} \subset e_2$. 
To prove the correspondence between the knowledge operator and relational semantics, we define transformations between the two:

\[ K : (\text{State} \rightarrow \text{State} \rightarrow \text{Set}) \rightarrow (\text{Event} \rightarrow \text{Event}) \]

\[ K[R] = \lambda e. \lambda w. \forall v. w R v \rightarrow e v \]

\[ R[K] : (\text{Event} \rightarrow \text{Event}) \rightarrow (\text{State} \rightarrow \text{State} \rightarrow \text{Set}) \]

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The fact that applying \( K[R] \) to an equivalence relation results in an operator satisfying S5 is well-known in the literature. We additionally had to prove preservation of semantic entailment. The inverse proof requires use of Axiom 5 and classical logic to show symmetry.
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To prove that the transformations are in fact inverse, we must show:

\[ K_{[R_K]} e w \leftrightarrow K e w \quad \text{and} \quad R_{[K_R]} w v \leftrightarrow R w v \]
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\[ \text{KFam}^w : (\Sigma e. K e w) \rightarrow \text{Event} \]
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By unfolding the definitions of the transformations, we find that:

\[ K_{[R_{[K]}]} e \rightarrow \prod (KFam^w) \subset e \]
We can replace the premise of the fourth direction using the previous fact, leaving us to prove $K e w$ from the assumption:

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Applying preservation of semantic entailment and instantiating at state $w$:

$$\prod(K \cdot K\text{Fam}^w) \cdot w \rightarrow K \cdot e \cdot w$$
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- We can conclude this by applying Axiom 4 to $h$. 
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From this point on, we postulate a non-empty set of agents and equip each $a \in \text{Agent}$ with their own knowledge relation $\simeq_a$. This also provides each with a knowledge operator $K_a = K_{[\simeq_a]}$. 
Common knowledge is defined in the relational semantics by its own equivalence relation, which we write $\propto$:

\begin{align*}
\text{Inductive } \propto: & \quad \text{State} \to \text{State} \to \text{Set} \\
\propto^- \text{union:} & \quad \forall a. \forall w. \forall v. w \equiv a v \to w \propto v \\
\propto^- \text{trans:} & \quad \forall w. \forall v. \forall u. w \propto v \to v \propto u \to w \propto u
\end{align*}

This is the transitive closure of the union of all agents' knowledge relations. It can be proved to be an equivalence relation using the fact that each $\equiv a$ is itself an equivalence relation.

Since it is an equivalence relation, we can generate a common knowledge operator from it:

$$r_{\text{CK}} : \text{Event} \to \text{Event}$$

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$$rCK : Event \rightarrow Event$$

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Defining a universal knowledge operator “everyone knows”:

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Coinductive Common Knowledge

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Following the informal description, we can see common knowledge of an event \( e \) as the infinite conjunction:

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\]

This leads naturally to a coinductive definition:

\[
\text{CoInductive} \ c\text{CK} : \text{Event} \rightarrow \text{Event} \\
c\text{CK} - \text{intro} : \text{EK} e \sqcap c\text{CK} (\text{EK} e) \subset c\text{CK} e
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However, we still need to establish that cCK is in fact equivalent to the relational common knowledge operator. That is, for all events $e$:

$$rCK \ e \equiv \ cCK \ e$$
We will only show the left-to-right direction here.
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- $rCK \ e \subset rCK \ (EK \ e)$

The proofs of these are by fully unfolding the definitions of $rCK$, $EK$, and $K_a$ until we are working directly with the underlying relations:

- $\forall w. (\forall v. w \propto v \rightarrow e v) \rightarrow \forall a. \forall u. w \simeq_a u \rightarrow e u$

- $\forall w. (\forall v. w \propto v \rightarrow e v) \rightarrow \forall u. w \propto u \rightarrow \forall a. \forall t. u \simeq_a t \rightarrow e t$
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- \( \forall w. (\forall v. w \propto v \rightarrow e v) \rightarrow \forall u. w \propto u \rightarrow \forall a. \forall t. u \simeq_a t \rightarrow e t \)

Then the assumptions can be combined with the constructors of \( \propto \) to reach the desired conclusions.
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- The statement we are to proving is: $\forall e. r_{CK} e \subseteq c_{CK} e$. 

  For $E_{CK} e$, we simply use the first of the previous lemmas with our assumption $r_{CK} e$.

  For $c_{CK} (E_{CK} e)$, we use the second of the previous lemmas, deriving $r_{CK} (E_{CK} e)$.

  We can then instantiate our coinductive hypothesis with the event $E_{CK} e$, and apply it to the intermediate result above, concluding $c_{CK} (E_{CK} e)$. 
The rest of the proof proceeds by coinduction:

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The rest of the proof proceeds by coinduction:

- The statement we are to proving is: \( \forall e. rCK \ e \subset cCK \ e \). We are allowed to assume it as a coinductive hypothesis provided that we use it only when guarded by constructor \( cCK \text{--intro} \).

- We assume \( rCK \ e \) and apply \( cCK \text{--intro} \), leaving us with two proof obligations: \( EK \ e \) and \( cCK (EK \ e) \).
The rest of the proof proceeds by coinduction:

- The statement we are to proving is: $\forall e. rCK\ e \subset cCK\ e$. We are allowed to assume it as a coinductive hypothesis provided that we use it only when guarded by constructor $cCK\!-\!intro$.
- We assume $rCK\ e$ and apply $cCK\!-\!intro$, leaving us with two proof obligations: $EK\ e$ and $cCK\ (EK\ e)$.
- For $EK\ e$, we simply use the first of the previous lemmas with our assumption $rCK\ e$. 

The rest of the proof proceeds by coinduction:

- The statement we are to proving is: $\forall e. \text{rCK } e \subseteq \text{cCK } e$. We are allowed to assume it as a coinductive hypothesis provided that we use it only when guarded by constructor $\text{cCK} – \text{intro}$.  
- We assume $\text{rCK } e$ and apply $\text{cCK} – \text{intro}$, leaving us with two proof obligations: $\text{EK } e$ and $\text{cCK } (\text{EK } e)$. 
- For $\text{EK } e$, we simply use the first of the previous lemmas with our assumption $\text{rCK } e$. 
- For $\text{cCK } (\text{EK } e)$, we use the second of the previous lemmas, deriving $\text{rCK } (\text{EK } e)$.
The rest of the proof proceeds by coinduction:

- The statement we are to proving is: $\forall e. rCK\ e \subseteq cCK\ e$. We are allowed to assume it as a coinductive hypothesis provided that we use it only when guarded by constructor $cCK\leftarrow intro$.

- We assume $rCK\ e$ and apply $cCK\leftarrow intro$, leaving us with two proof obligations: $EK\ e$ and $cCK\ (EK\ e)$.

- For $EK\ e$, we simply use the first of the previous lemmas with our assumption $rCK\ e$.

- For $cCK\ (EK\ e)$, we use the second of the previous lemmas, deriving $rCK\ (EK\ e)$.

- We can then instantiate our coinductive hypothesis with the event $EK\ e$, and apply it to the intermediate result above, concluding $cCK\ (EK\ e)$. 
Conclusion

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